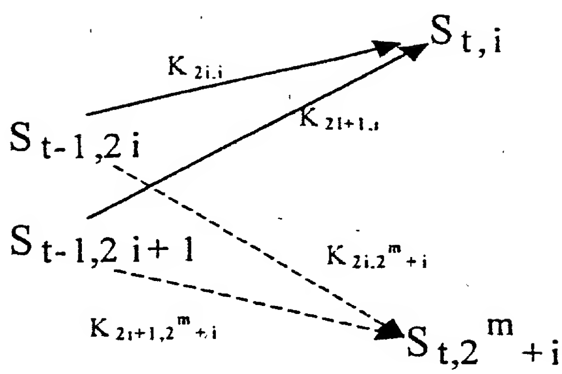

Figure 1A Transfer between states

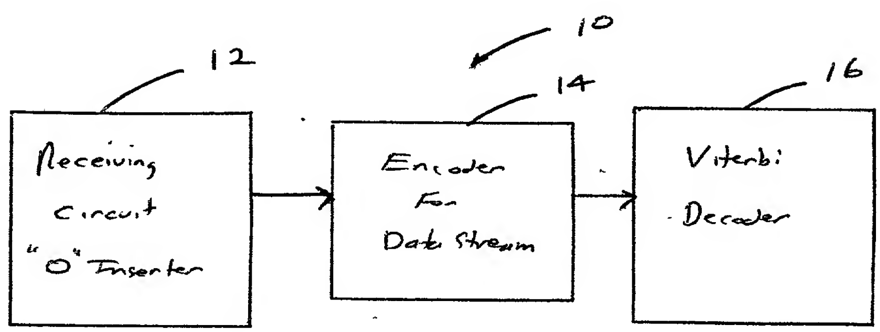

FIG. 1B

$$
\begin{aligned}
s_{t,i} &= D^{k_{2i,i}} s_{t-1,2i} + D^{k_{2i+1,i}} s_{t-1,2i+1}, \\
s_{t,2^m+i} &= D^{k_{2i,2^m+i}} s_{t-1,2i} + D^{k_{2i+1,2^m+i}} s_{t-1,2i+1}.
\end{aligned}
$$

Thus, let $a_{2^{m-1}} = D^{k_{1,2^{m-1}}}$ but

$$a_j = \left[D^{k_{2j,j}}, D^{k_{2j+1,j}}\right], j \neq 2^{m-1},$$

as a result:

$$S_t = \begin{bmatrix} s_{t,1} \\ \vdots \\ s_{t,2^m-1} \end{bmatrix} = TS_{t-1}, \quad S_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ D^{k_{0,2^{m-1}}} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where

$$T = \begin{bmatrix}
0 & a_1 & 0 & \cdots & 0 \\
0 & 0 & a_2 & \cdots & 0 \\
. & . & . & \cdots & . \\
0 & 0 & 0 & \cdots & a_{2^{m-1}-1} \\
a_{2^{m-1}} & 0 & 0 & \cdots & 0 \\
0 & a_{2^{m-1}+1} & 0 & \cdots & 0 \\
0 & 0 & a_{2^{m-1}+2} & \cdots & 0 \\
. & . & . & \cdots & . \\
0 & 0 & 0 & \cdots & a_{2^m-1}
\end{bmatrix}.$$

Now, let

$$\vec{\xi} = \sum_{t=0}^{\infty} S_t, \qquad (1)$$

then

$$\vec{\xi} = \sum_{t=0}^{\infty} T^t S_0 = (I - T)^{-1} S_0.$$

FIG. 2

$$s_{t,0} = D^{k_{1,0}} s_{t-1,1}, \text{ let}$$

$$\begin{aligned} \xi_0 &= \sum_{t=0}^{\infty} s_{t,0} = \left[\begin{array}{cccc} D^{k_{1,0}} & 0 & \ldots & 0 \end{array} \right] \overrightarrow{\xi} \\ &= \left[\begin{array}{cccc} D^{k_{1,0}} & 0 & \ldots & 0 \end{array} \right] (I-T)^{-1} S_0. \quad (2) \end{aligned}$$

FiG. 3

$$T_0 = \begin{bmatrix} 0 & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_2 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & a_{2^{m-1}-1} \\ 0 & 0 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

we have

$$S_t = T_0 S_{t-1}.$$

Assume that the zero is inserted at the $j_0 - th$ position after the error one happens. Then

$$S_{nK+j} = \begin{cases} T^j S_{nK}, & 0 \le j < j_0 \\ T^{j-j_0} T_0 T^{j_0-1} S_{nK}, & j_0 \le j < K. \end{cases}$$

Now, let

$$\begin{aligned} P &= I + \dots + T^{j_0-1} + T_0 T^{j_0-1} \\ &\quad + \dots + T^{K-j_0} T_0 T^{j_0-1}, \\ T_K &= T^{K-j_0} T_0 T^{j_0-1}, \end{aligned}$$

we have

$$\begin{aligned} S_{nK} &= T_K S_{(n-1)K}, \\ \vec{\xi} &= \sum_{n=0}^{\infty} \sum_{j=0}^{K-1} S_{nK+j} = \sum_{n=0}^{\infty} P S_{nK} \\ &= P \sum_{n=0}^{\infty} T_K^n S_0 = P (I - T_K)^{-1} S_0 \end{aligned}$$

FIG. 4

$$T = \begin{bmatrix} 0 & D & D \\ 1 & 0 & 0 \\ 0 & D & D \end{bmatrix}.$$

as a result:

$$[D^2, 0, 0]\,(I - T)^{-1} \begin{bmatrix} 0 \\ D^2 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{vmatrix} 0 & -D & -D \\ 1 & 1 & 0 \\ 0 & -D & 1-D \end{vmatrix}}{\begin{vmatrix} 1 & -D & -D \\ -1 & 1 & 0 \\ 0 & -D & 1-D \end{vmatrix}} D^4$$

$$= \frac{D^5}{1-2D}.$$

FIG. 5

$$P = \begin{bmatrix} 1 & D & D \\ 0 & 1+D & D \\ 0 & 0 & 1 \end{bmatrix},$$

$$T_K = \begin{bmatrix} 0 & D^2 & D^2 \\ 0 & 0 & 0 \\ 0 & D^2 & D^2 \end{bmatrix},$$

$$\begin{aligned} \xi_0 &= [1, D, D] \begin{bmatrix} 1 & -D^2 & -D^2 \\ 0 & 1 & 0 \\ 0 & -D^2 & 1-D^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} D^4 \\ &= \frac{\begin{vmatrix} 1 & -D^2 & -D^2 \\ 1 & D & D \\ 0 & -D^2 & 1-D^2 \end{vmatrix}}{\begin{vmatrix} 1 & -D^2 & -D^2 \\ 0 & 1 & 0 \\ 0 & -D^2 & 1-D^2 \end{vmatrix}} D^4 \\ &= \frac{D^5}{1-D} = D^5 \sum_{k=0}^{\infty} D^k. \end{aligned}$$

FIG. 6

$$P = \begin{bmatrix} 1+D & D+D^2 & D+D^2 \\ 1 & 1 & 0 \\ 0 & D & 1+D \end{bmatrix},$$

$$T_K = \begin{bmatrix} 0 & 0 & 0 \\ D & D^2 & D^2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \xi_0 &= [1, D, D] \times \\ & \quad \begin{bmatrix} 1 & 0 & 0 \\ -D & 1-D^2 & -D^2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} (1+D)\, D^4 \\ &= \frac{\begin{vmatrix} 1 & 0 & 0 \\ 1 & D & D \\ 0 & 0 & 1 \end{vmatrix} (1+D)\, D^4}{\begin{vmatrix} 1 & 0 & 0 \\ -D & 1-D^2 & -D^2 \\ 0 & 0 & 1 \end{vmatrix}} \\ &= \frac{D^5}{1-D} = D^5 \sum_{k=0}^{\infty} D^k. \end{aligned}$$

FIG. 7